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FOURIER EXPANSIONS OF FUNCTIONS WITH BOUNDED VARIATION OF SEVERAL VARIABLES

LEONARDO COLZANI

Abstract. In the first part of the paper we establish the pointwise convergence as $t \to +\infty$ for convolution operators $\int_{\mathbb{R}^d} t^d K(ty)\,\varphi(x-y)dy$ under the assumptions that $\varphi(y)$ has integrable derivatives up to an order α and that $|K(y)| \le c\,(1+|y|)^{-\beta}$ with $\alpha+\beta>d$. We also estimate the Hausdorff dimension of the set where divergence may occur. In particular, when the kernel is the Fourier transform of a bounded set in the plane, we recover a two-dimensional analog of the Dirichlet theorem on the convergence of Fourier series of functions with bounded variation. In the second part of the paper we prove an equiconvergence result between Fourier integrals on euclidean spaces and expansions in eigenfunctions of elliptic operators on manifolds, which allows us to transfer some of the results proved for Fourier integrals to eigenfunction expansions. Finally, we present some examples of different behaviors between Fourier integrals, Fourier series and spherical harmonic expansions.

Fourier integrals on Euclidean spaces

This section is devoted to the study of the pointwise convergence as $t \to +\infty$ for operators of the type

$$\int_{\mathbb{R}^d} \chi\left(t^{-1}\xi\right) \widehat{\varphi}(\xi) \exp(2\pi i \xi x) d\xi = \int_{\mathbb{R}^d} t^d K\left(ty\right) \varphi(x-y) dy.$$

Here $\varphi(x)$ is an integrable function, $\widehat{\varphi}(\xi) = \int_{\mathbb{R}^d} \varphi(x) \exp(-2\pi i \xi x) dx$ is its Fourier transform, $\chi(\xi)$ is an integrable Fourier multiplier and $K(x) = \int_{\mathbb{R}^d} \chi(\xi) \exp(2\pi i \xi x) d\xi$ is the associated convolution kernel. For example, when the multiplier is the characteristic function of a bounded domain which contains the origin, the problem reduces to the pointwise inversion of Fourier transform,

$$\lim_{t\to+\infty} \int_{t\Omega} \widehat{\varphi}(\xi) \exp(2\pi i \xi x) d\xi = \varphi(x).$$

A classical reference for this problem is [3]. Other references more directly related to this paper are [1], [4], [5], [6], [15], [16], [19], [23], [24], [25] which are devoted to Fourier expansions of piecewise smooth functions, [2] and [22], which estimate the capacity of the divergence sets of one-dimensional Fourier series, [7], [12], [17], [26], which contain results on spherical summability of Fourier integrals of functions in Sobolev classes, [18], with a simple proof of the almost everywhere convergence of expansions in eigenfunctions of functions in Sobolev classes. Finally, a special mention should be made to the research tutorial [21]. In order to motivate what

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follows, let us consider two extreme cases. If the function $\varphi(x)$ has integrable derivatives up to an order $\alpha > d$, then $|\widehat{\varphi}(\xi)| \leq c\,(1+|\xi|)^{-\alpha}$ is integrable. Hence, if $\chi\,(\xi)$ is bounded and continuous at the origin, then

$$\lim_{t\to+\infty}\int_{\mathbb{R}^d}\chi\left(t^{-1}\xi\right)\widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi = \chi\,(0)\,\varphi(x).$$

If the kernel $K(x)$ has the decay $|K(x)| \leq c\,(1+|x|)^{-\beta}$ with $\beta > d$, then $K(x)$ is integrable. Hence, if $\varphi(y)$ is bounded and continuous at the point x, then

$$\lim_{t\to+\infty}\int_{\mathbb{R}^d} t^d K\,(ty)\,\varphi(x-y)dy = \varphi(x)\int_{\mathbb{R}^d} K\,(y)\,dy.$$

Here we want to consider an intermediate case between these two extremes and, roughly speaking, our result is that α derivatives of the function expanded and a decay of order β of the summation kernel with $\alpha+\beta > d$ are sufficient conditions for the pointwise inversion of the Fourier transform, with a possible exception of a set of points with Hausdorff dimension at most $d-\alpha$. In what follows the smoothness of a function is measured with the Riesz potentials or fractional powers $\Delta^{\alpha/2}$ of the Laplace operator $\Delta = -\sum_j \partial^2/\partial x_j^2$, defined spectrally by $\widehat{\Delta^{\alpha/2}\varphi}(\xi) = (2\pi\,|\xi|)^{\alpha}\,\widehat{\varphi}(\xi)$. However, instead of $\Delta^{1/2}$ one can also use the gradient $\nabla = \{\partial/\partial x_j\}$. Here it is our basic result.

Theorem 1. *Let $\chi(\xi)$ be an integrable Fourier multiplier, smooth in a neighborhood of the origin with $\chi(0) = 1$. Assume that the associated convolution kernel $K(y)$ has decay $|K(y)| \leq c\,(1+|y|)^{-\beta}$. Finally, let $\varphi(y)$ be an integrable function and assume that $\Delta^{\alpha/2}\varphi(y)$ is a locally finite measure. Then for every $\gamma > 0$,*

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\chi\left(t^{-1}\xi\right)\widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi - \varphi(x)\right| \\
&= \left|\int_{\mathbb{R}^d} t^d K\,(ty)\,\varphi(x-y)dy - \varphi(x)\right| \\
&\leq c\int_{\mathbb{R}^d} t^d\,(1+t\,|y|)^{-\gamma}\,|\varphi(x-y)-\varphi(x)|\,dy \\
&+c\int_{\mathbb{R}^d} t^{d-\alpha}\,(1+t\,|y|)^{-\beta}\left|\Delta^{\alpha/2}\varphi(x-y)\right|dy.
\end{aligned}$$

Observe that if $\chi(\xi)$ and $\varphi(y)$ are integrable, then $K\,(y)$ and $\widehat{\varphi}(\xi)$ are bounded and this guarantees the existence of the integrals

$$\int_{\mathbb{R}^d}\chi\left(t^{-1}\xi\right)\widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi, \qquad \int_{\mathbb{R}^d} t^d K\,(ty)\,\varphi(x-y)dy.$$

However, this integrability assumption on $\chi(\xi)$ and $\varphi(y)$ can be weakened, as soon as the above two integrals are well defined. It is not difficult to convert the theorem into a convergence result.

Theorem 2. *Let $\chi(\xi)$ be an integrable Fourier multiplier, smooth in a neighborhood of the origin with $\chi(0) = 1$. Assume that the associated convolution kernel $K(y)$ has decay $|K(y)| \leq c\,(1+|y|)^{-\beta}$. Finally, let $\varphi(y)$ be an integrable function and assume that $(1+|y|)^{\alpha-d}\,\Delta^{\alpha/2}\varphi(y)$ is a finite measure. If $\alpha+\beta > d$, then the convergence*

$$\lim_{t\to+\infty}\int_{\mathbb{R}^d} t^d K\,(ty)\,\varphi(x-y)dy = \varphi(x)$$

holds with the possible exception of a set of points with Hausdorff dimension at most $d-\alpha$. More precisely, assume that at a point x there exists $0<\varepsilon<1$ such that when $r\to 0+$,

$$\int_{\{|y|\le r\}}\left|\Delta^{\alpha/2}\varphi(x-y)\right|dy\le cr^{d-\alpha+\varepsilon}.$$

Then, when $t\to+\infty$,

$$\left|\int_{\mathbb{R}^d}t^dK\left(ty\right)\varphi(x-y)dy-\varphi(x)\right|\le\begin{cases}ct^{-\varepsilon} & \text{if } \varepsilon<\alpha+\beta-d,\\ ct^{-\varepsilon}\log(t) & \text{if } \varepsilon=\alpha+\beta-d,\\ ct^{d-\alpha-\beta} & \text{if } \varepsilon>\alpha+\beta-d.\end{cases}$$

A by product of the above result is another proof of the well-known fact that the singularities of functions in Sobolev classes of index α have Hausdorff dimension at most $d-\alpha$. On the other hand, there are functions in these spaces which are infinite on sets of dimension $d-\alpha$. Hence the estimate of the dimension of the divergence set is best possible. Later we will show that also the condition $\alpha+\beta>d$ is best possible. By substituting the square root of the Laplace operator with the gradient, one can apply the theorem to functions with bounded variation, that is, integrable functions whose distributional first derivatives are finite measures. In particular, the theorems apply to piecewise smooth functions, which are linear combinations of functions $\psi(x)\chi_D(x)$, products of smooth functions and characteristic functions of bounded domains with smooth boundary. The content of the following theorem is that in this case convergence holds everywhere, even along the discontinuities of the functions expanded.

Theorem 3. *Let $\chi(\xi)$ be an integrable Fourier multiplier, with compact support, smooth in a neighborhood of the origin and with $\chi(0)=1$. Assume that the associated convolution kernel $K(y)$ has decay $|K(y)|\le c\left(1+|y|\right)^{-\beta}$ for some $\beta>d-1$. If $\varphi(y)$ is a piecewise smooth function suitably normalized along the discontinuities, then at every point x,*

$$\lim_{t\to+\infty}\int_{\mathbb{R}^d}\chi\left(t^{-1}\xi\right)\widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi=\lim_{t\to+\infty}\int_{\mathbb{R}^d}t^dK\left(ty\right)\varphi(x-y)dy=\varphi(x).$$

Classical examples of Fourier multipliers to which the above theorems apply are the Bochner-Riesz multipliers.

Corollary 4. *The Bochner-Riesz multipliers $(1-|\xi|^2)_+^\gamma$ are associated to the kernels $\pi^{-\gamma}\Gamma(\gamma+1)\,|y|^{-\gamma-d/2}\,J_{\gamma+d/2}\left(2\pi\,|y|\right)$, which have decay $c\left(1+|y|\right)^{-\gamma-(d+1)/2}$. The Bochner-Riesz means with index $\gamma>(d-1)/2-\alpha$ of functions with α integrable derivatives converge, with a possible exception of a set of points with Hausdorff dimension at most $d-\alpha$,*

$$\lim_{t\to+\infty}\int_{\{|\xi|<t\}}\left(1-\left|t^{-1}\xi\right|^2\right)^\gamma\widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi=\varphi(x).$$

Other examples of Fourier multipliers to which the theorems apply are the characteristic functions of bounded domains. Indeed, the Fourier transform of a domain with boundary of finite Minkowski measure has decay $|\widehat{\chi}_\Omega(y)|\le c\left(1+|y|\right)^{-1}$. If the boundary is analytic, the exponent -1 can be replaced by $-\beta$ for some $\beta>1$. If the domain is convex with a smooth boundary with positive Gauss curvature, the exponent becomes $-(d+1)/2$. In particular, in dimension one when the domain is an interval, one recaptures the Dirichlet theorem on the convergence of Fourier integrals of functions with bounded variation, and in dimension two, if the domain has just a bit of smoothness, one obtains a generalization of this Dirichlet theorem.

Corollary 5. *Let Ω be a bounded planar domain containing the origin and assume that $|\widehat{\chi}_\Omega(y)| \leq c\,(1+|y|)^{-\beta}$ for some $\beta > 1$. If $\varphi(y)$ is an integrable function with bounded variation, then, with a possible exception of a set of points with Hausdorff dimension at most* 1*,*

$$\lim_{t \to +\infty} \int_{t\Omega} \widehat{\varphi}(\xi) \exp(2\pi i \xi x) d\xi = \varphi(x).$$

Moreover, if $\varphi(y)$ is piecewise smooth, then convergence holds at every point.

The above corollaries are essentially already known and they are included here only because they easily follow from our main results and have been the original motivation for our work. In particular, Corollary 4 for a restricted range of indexes is already contained in [12] and [26], while Corollary 5 for spherical summability is already contained in [21], Sections 3, 4, 5, and Propositions 4 and 8. General partial sums of Fourier integrals of piecewise smooth functions are discussed in [1] and [6] and the associated Gibbs phenomenon in [4], [8], [9], [10], [11], and [25]. An example of domain which does not satisfy the assumptions of Corollary 5 is a domain with flat boundary, since $\widehat{\chi}_\Omega(y)$ has a decay $(1+|y|)^{-1}$ if $y \to \infty$ along directions perpendicular to flat parts of the boundary. In this case convergence of Fourier expansion may fail. Since the uniform decay of Fourier transforms of characteristic functions is never better than $c\,(1+|y|)^{-(d+1)/2}$, the corollary does not apply to functions with bounded variation in dimensions greater than two. Indeed, in dimension $d \geq 3$, the partial sums of Fourier integrals of piecewise smooth function may not converge, even at points where the functions expanded are smooth; see [20] and [16] or the examples in the last section of the paper. In the second section of the paper we shall give an extension of Corollary 4 to expansions in eigenfunctions of elliptic operators, for a restricted range of indexes, and in the final section we shall present some examples of different behaviors between Fourier integrals, Fourier series and spherical harmonic expansions.

Proof of Theorem 1. If $\chi(\xi)$ is smooth around the origin, there exists a smooth cutoff $\psi(\xi)$ identically one in a neighborhood of the origin and zero in a slightly larger neighborhood such that $\psi(\xi)\chi(\xi)$ is smooth everywhere. Denoting by $t^d K(ty)$ $= t^d A(ty) + t^d B(ty)$ the convolution kernels associated to the dilated multipliers $\chi\left(t^{-1}\xi\right) = \psi\left(t^{-1}\xi\right)\chi\left(t^{-1}\xi\right) + \left(1-\psi\left(t^{-1}\xi\right)\right)\chi\left(t^{-1}\xi\right)$, one obtains the decomposition

$$\begin{aligned}\int_{\mathbb{R}^d} \chi\left(t^{-1}\xi\right) \widehat{\varphi}(\xi) \exp(2\pi i \xi x) d\xi &= \int_{\mathbb{R}^d} t^d K(ty)\varphi(x-y)dy \\ &= \int_{\mathbb{R}^d} t^d A(ty)\varphi(x-y)dy + \int_{\mathbb{R}^d} t^d B(ty)\varphi(x-y)dy.\end{aligned}$$

By our assumptions $\psi(\xi)\chi(\xi)$ is smooth with compact support and the associated kernel has a fast decay at infinity, $|A(y)| \leq c\,(1+|y|)^{-\gamma}$ for every γ. Moreover,

$\int_{\mathbb{R}^d} A(y)dy = \psi(0)\,\chi(0) = 1$. Hence,

$$\left|\int_{\mathbb{R}^d} t^d A(ty)\,\varphi(x-y)dy - \varphi(x)\right| = \left|\int_{\mathbb{R}^d} t^d A(ty)\,(\varphi(x-y)-\varphi(x))\,dy\right|$$
$$\leq c\int_{\mathbb{R}^d} t^d\,(1+t\,|y|)^{-\gamma}\,|\varphi(x-y)-\varphi(x)|\,dy.$$

In order to estimate the convolution with the kernel $B(y)$, write

$$\int_{\mathbb{R}^d} t^d B(ty)\varphi(x-y)dy = \int_{\mathbb{R}^d} \Delta^{-\alpha/2}\left(t^d B(ty)\right)\Delta^{\alpha/2}\varphi(x-y)dy.$$

The crucial observation is that $\Delta^{-\alpha/2}\left(t^d B(ty)\right) = t^{d-\alpha}C(ty)$ is $t^{-\alpha}$ better than $t^d B(ty)$,

$$\Delta^{-\alpha/2}\left(t^d B(ty)\right) = \int_{\mathbb{R}^d} (2\pi\,|\xi|)^{-\alpha}\left(1-\psi\left(t^{-1}\xi\right)\right)\chi\left(t^{-1}\xi\right)\exp(2\pi i\xi y)d\xi$$
$$= t^{d-\alpha}\int_{\mathbb{R}^d} (2\pi\,|\zeta|)^{-\alpha}\,(1-\psi(\zeta))\,\chi(\zeta)\exp(2\pi i t y\zeta)d\zeta.$$

Moreover, the kernel $C(y)$ has at least the same decay at infinity as $K(y)$. Indeed, $C(y)$ is the convolution of the kernels $D(y)$ and $K(y)$ associated to the multipliers $(2\pi\,|\zeta|)^{-\alpha}\,(1-\psi(\zeta))$ and $\chi(\zeta)$. Since by our assumptions $|K(y)| \leq c\,(1+|y|)^{-\beta}$ and since $D(y)$ behaves like $|y|^{\alpha-d}$ at the origin and decays rapidly at infinity, $|D(y)| \leq c\,|y|^{\alpha-d}\,(1+|y|)^{-\gamma}$ for every γ,

$$|C(y)| = \left|\int_{\mathbb{R}^d} D(z)K(y-z)dz\right|$$
$$\leq c\int_{\mathbb{R}^d} |z|^{\alpha-d}\,(1+|z|)^{-\gamma}\,(1+|y-z|)^{-\beta}\,dz \leq c\,(1+|y|)^{-\beta}.$$

Hence,

$$\left|\int_{\mathbb{R}^d} t^{d-\alpha}C(ty)\Delta^{\alpha/2}\varphi(x-y)dy\right| \leq c\int_{\mathbb{R}^d} t^{d-\alpha}\,(1+t\,|y|)^{-\beta}\left|\Delta^{\alpha/2}\varphi(x-y)\right|dy. \quad \square$$

For the proof of Theorem 2 we will need the following properties of integral averages of Riesz potentials. The idea behind this is the general principle that the pointwise behavior of functions in Sobolev classes is somehow better than the one of merely integrable functions.

Lemma 6. *Let $(1+|y|)^{\alpha-d}\,\mu(y)$, with $0<\alpha<d$, be a finite measure and assume that at a point x there exists $0<\varepsilon<1$ such that when $r\to 0+$,*

$$\int_{\{|y|\leq r\}} |\mu(x-y)|\,dy \leq cr^{d-\alpha+\varepsilon}.$$

Then x is a Lebesgue point for the Riesz potential $\Delta^{-\alpha/2}\mu(y)$ and

$$\int_{\{|y|\leq r\}} \left|\Delta^{-\alpha/2}\mu(x-y) - \Delta^{-\alpha/2}\mu(x)\right|dy \leq cr^{d+\varepsilon}.$$

Proof of Lemma 6. By rotation invariance and homogeneity the Riesz potential $\Delta^{-\alpha/2}$ is given, up to a suitable constant, by the convolution with the kernel

$|y|^{\alpha-d}$. If $\psi(y)$ is a smooth cutoff identically one in $\{|y| \leq 1\}$ and identically zero in $\{|y| \geq 2\}$, then

$$\int_{\mathbb{R}^d} |y|^{\alpha-d} \mu(x-y)dy = \int_{\mathbb{R}^d} (1-\psi(y))\, |y|^{\alpha-d} \mu(x-y)dy + \int_{\mathbb{R}^d} \psi(y)\, |y|^{\alpha-d} \mu(x-y)dy.$$

The kernel $(1-\psi(y))\, |y|^{\alpha-d}$ is smooth and, by the assumptions on $\mu(y)$, the second integral is absolutely convergent and defines a smooth function. It then suffices to prove that x is a Lebesgue point of the third convolution,

$$\begin{aligned}
&\int_{\{|z|\leq r\}} \left| \int_{\mathbb{R}^d} \psi(y)\, |y|^{\alpha-d} \mu(x-y-z)dy - \int_{\mathbb{R}^d} \psi(y)\, |y|^{\alpha-d} \mu(x-y)dy \right| dz \\
&\quad \leq \int_{\mathbb{R}^d} \left(\int_{\{|z|\leq r\}} \left| \psi(y-z)\, |y-z|^{\alpha-d} - \psi(y)\, |y|^{\alpha-d} \right| dz \right) |\mu(x-y)|\, dy \\
&\leq cr^d \int_{\{|y|\leq r\}} |y|^{\alpha-d} |\mu(x-y)|\, dy + cr^{d+1} \int_{\{r\leq |y| \leq r+2\}} |y|^{\alpha-d-1} |\mu(x-y)|\, dy.
\end{aligned}$$

Under the assumption that $\int_{\{|y|\leq r\}} |\mu(x-y)|\, dy \leq cr^{d-\alpha+\varepsilon}$, the last two integrals are dominated by $r^{d+\varepsilon}$. □

Lemma 7. *If $\mu(y)$ is a locally finite measure and $0 \leq \gamma \leq d$, then*

$$\limsup_{r\to 0+} r^{-\gamma} \int_{\{|y|\leq r\}} |\mu(x-y)|\, dy < +\infty$$

with the possible exception of a set of points with Hausdorff dimension at most γ.

Proof of Lemma 7. Fix $\delta, k > 0$ and define the set of points

$$A(\delta,k) = \left\{ |x| < k,\ \limsup_{r\to 0+} r^{-\gamma} \int_{\{|y-x|\leq r\}} |\mu(y)|\, dy > \delta \right\}.$$

Given $R > 0$, the family of balls with center $x \in A(\delta,k)$ and radius $r \leq R$ and such that $\int_{\{|y-x|\leq r\}} |\mu(y)|\, dy > \delta r^\gamma$ covers $A(\delta,k)$. By a standard argument from this family one can extract disjoint balls with $A(\delta,k) \subseteq \bigcup \{|y-x| \leq 5r\}$ and summing over this subfamily one obtains

$$\sum (10r)^\gamma \leq 10^\gamma \delta^{-1} \sum \int_{\{|y-x|\leq r\}} |\mu(y)|\, dy \leq 10^\gamma \delta^{-1} \int_{\{|y|\leq k+R\}} |\mu(y)|\, dy.$$

Letting $R \to 0+$ one deduces that the γ-dimensional Hausdorff measure of $A(\delta,k)$ is finite. □

Proof of Theorem 2. By Theorem 1,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} t^d K(ty)\, \varphi(x-y)dy - \varphi(x) \right| \\
&\leq c \int_{\mathbb{R}^d} t^d (1+t|y|)^{-\gamma} |\varphi(x-y) - \varphi(x)|\, dy \\
&+ c \int_{\mathbb{R}^d} t^{d-\alpha} (1+t|y|)^{-\beta} \left| \Delta^{\alpha/2} \varphi(x-y) \right| dy.
\end{aligned}$$

Assume that $\int_{\{|y|\le r\}} \left|\Delta^{\alpha/2}\varphi(x-y)\right| dy \le cr^{d-\alpha+\varepsilon}$. Then, by Lemma 6, $\int_{\{|y|\le r\}} |\varphi(x-y)-\varphi(x)|\, dy \le cr^{d+\varepsilon}$ and an integration by parts gives

$$\begin{aligned}
&\int_{\mathbb{R}^d} t^d \left(1+t\,|y|\right)^{-\gamma} |\varphi(x-y)-\varphi(x)|\, dy \\
&= \gamma t^{d+1} \int_0^{+\infty} \left(1+tr\right)^{-\gamma-1} \int_{\{|y|\le r\}} |\varphi(x-y)-\varphi(x)|\, dy dr \\
&\le ct^{d+1} \int_0^{+\infty} r^{d+\varepsilon} \left(1+tr\right)^{-\gamma-1} dr \le ct^{-\varepsilon}.
\end{aligned}$$

Again, if $\int_{\{|y|\le r\}} \left|\Delta^{\alpha/2}\varphi(x-y)\right| dy \le cr^{d-\alpha+\varepsilon}$, then

$$\begin{aligned}
&\int_{\{|y|\le 1\}} t^{d-\alpha} \left(1+t\,|y|\right)^{-\beta} \left|\Delta^{\alpha/2}\varphi(x-y)\right| dy \\
&\le t^{d-\alpha} \int_{\{|y|\le t^{-1}\}} \left|\Delta^{\alpha/2}\varphi(x-y)\right| dy \\
&+ t^{d-\alpha} \sum_{k=0}^{[\log_2(t)]} 2^{-\beta k} \int_{\{t^{-1}2^k\le |y|\le t^{-1}2^{k+1}\}} \left|\Delta^{\alpha/2}\varphi(x-y)\right| dy \\
&\le ct^{-\varepsilon} + ct^{-\varepsilon} \sum_{k=0}^{[\log_2(t)]} 2^{(d-\alpha-\beta+\varepsilon)k} \le \begin{cases} ct^{-\varepsilon} & \text{if } \varepsilon<\alpha+\beta-d, \\ ct^{-\varepsilon}\log(t) & \text{if } \varepsilon=\alpha+\beta-d, \\ ct^{d-\alpha-\beta} & \text{if } \varepsilon>\alpha+\beta-d. \end{cases}
\end{aligned}$$

Moreover, if $\left(1+|y|\right)^{\alpha-d} \Delta^{\alpha/2}\varphi(y)$ is a finite measure and $\alpha+\beta>d$, then

$$\begin{aligned}
&\int_{\{|y|\ge 1\}} t^{d-\alpha} \left(1+t\,|y|\right)^{-\beta} \left|\Delta^{\alpha/2}\varphi(x-y)\right| dy \\
&\le ct^{d-\alpha-\beta} \int_{\mathbb{R}^d} \left(1+|y|\right)^{\alpha-d} \left|\Delta^{\alpha/2}\varphi(y)\right| dy.
\end{aligned}$$

Finally, by Lemma 7 the inequality $\int_{\{|y|\le r\}} \left|\Delta^{\alpha/2}\varphi(x-y)\right| dy \le cr^{d-\alpha+\varepsilon}$ holds with a possible exception of a set of points with Hausdorff dimension $d-\alpha+\varepsilon$. Letting $\varepsilon\to 0+$ we deduce that convergence holds with a possible exception of a set of points with Hausdorff dimension $d-\alpha$. □

An alternative way to prove Theorem 2 is to establish the boundedness of the maximal operator associated to the family of convolution operators. As in the proof of Theorem 1, one can write

$$\int_{\mathbb{R}^d} t^d K(ty)\varphi(x-y)dy = \int_{\mathbb{R}^d} t^{d-\alpha} H\left(ty\right) \Delta^{\alpha/2}\varphi(x-y)dy,$$

with

$$H(y) = \int_{\mathbb{R}^d} \left(2\pi\,|\xi|\right)^{-\alpha} \chi\left(\xi\right) \exp(2\pi i\xi y) d\xi.$$

Under our assumptions it can be proved that $|H(y)| \le c\left(1+|y|\right)^{\alpha-d}$, so that $\sup_{t>0} \left|t^{d-\alpha}H\left(ty\right)\right| \le c\,|y|^{\alpha-d}$ and

$$\sup_{t>0} \left| \int_{\mathbb{R}^d} t^d K(ty)\varphi(x-y)dy \right| \le c \int_{\mathbb{R}^d} |y|^{\alpha-d} \left|\Delta^{\alpha/2}\varphi(x-y)\right| dy.$$

This estimate suggests studying the set of divergence in terms of capacity. The approach followed in the above proofs of Theorem 1 and Theorem 2 is perhaps a bit more complicated, however, it has the advantage of giving a quite explicit description of the set of points where convergence takes place and an estimate of the speed of convergence.

Proof of Theorem 3. Let $\varphi(y)$ be piecewise smooth and let $\nabla\varphi(y)dy = \mu(y)dy + \nu(y)d\sigma(y)$ be the decomposition of its gradient into an absolutely continuous part $\mu(y)dy$ and a singular part $\nu(y)d\sigma(y)$ supported on a smooth surface S. Then, as in the proof of Theorem 1,

$$\begin{aligned}
&\int_{\mathbb{R}^d} t^d K(ty)\varphi(x-y)dy \\
&= \int_{\mathbb{R}^d} t^d A(ty)\varphi(x-y)dy - \int_{\mathbb{R}^d} \nabla\Delta^{-1}\left(t^d B(ty)\right)\nabla\varphi(x-y)dy \\
&= \int_{\mathbb{R}^d} t^d A(ty)\varphi(x-y)dy + \int_{\mathbb{R}^d} t^{d-1} C(ty)\mu(x-y)dy \\
&\quad + \int_S t^{d-1} C(tx-ty)\nu(y)d\sigma(y).
\end{aligned}$$

For a piecewise smooth function suitably normalized along the discontinuities, every point is Lebesgue and the absolutely continuous part of the gradient is bounded with compact support, so that

$$\lim_{t\to+\infty}\int_{\mathbb{R}^d} t^d A(ty)\varphi(x-y)dy = \varphi(x),$$
$$\lim_{t\to+\infty}\int_{\mathbb{R}^d} t^{d-1} C(ty)\mu(x-y)dy = 0.$$

It remains to consider the convolution between the vector-valued kernel $t^{d-1}C(ty)$ and the singular part of the gradient $\nu(y)d\sigma(y)$, but this is essentially a $(d-1)$-dimensional integral of a smooth function $\nu(y)$ against a $d-1$ integrable kernel $t^{d-1}C(tx-ty)$. If x is not on S, then

$$\lim_{t\to+\infty}\int_S t^{d-1} C(tx-ty)\nu(y)d\sigma(y) = 0.$$

If x is a point on S and $T(x)$ the hyperplane tangent to S at x, then

$$\lim_{t\to+\infty}\int_S t^{d-1} C(tx-ty)\nu(y)d\sigma(y) = \nu(x)\int_{T(x)} C(x-y)d\sigma(y).$$

Indeed,

$$\begin{aligned}
&\int_S t^{d-1} C(tx-ty)\nu(y)d\sigma(y) \\
&= \int_S t^{d-1} C(tx-ty)\left(\nu(y)-\nu(x)\right)d\sigma(y) + \nu(x)\int_{tx-tS} C(z)d\sigma(z).
\end{aligned}$$

Since $|\nu(y)-\nu(x)| \le c\,|y-x|$ and $|C(tx-ty)| \le c\,(1+t\,|x-y|)^{-\beta}$, if $d-1<\beta<d$, then

$$\left|\int_S t^{d-1} C(tx-ty)\left(\nu(y)-\nu(x)\right)d\sigma(y)\right| \le ct^{d-\beta-1}.$$

Since $|C(z)| \le c\,(1+|z|)^{-\beta}$ is integrable with respect to a $(d-1)$-dimensional surface measure, given $\varepsilon > 0$, there exists $R > 0$ such that

$$\left| \int_{\{tx - tS\} \cap \{|z| \ge R\}} C(z) d\sigma(z) \right| < \varepsilon, \qquad \left| \int_{\{tx - tT(x)\} \cap \{|z| \ge R\}} C(z) d\sigma(z) \right| < \varepsilon.$$

Now we use the assumption that $\chi\,(\xi)$ has compact support, which implies that $C(z)$ is smooth. Since $\{tx - tS\} \cap \{|z| \le R\} \to \{tx - tT(x)\} \cap \{|z| \le R\}$ as $t \to +\infty$, if t is large enough, then

$$\left| \int_{\{tx - tS\} \cap \{|z| \le R\}} C(z) d\sigma(z) - \int_{\{tx - tT(x)\} \cap \{|z| \le R\}} C(z) d\sigma(z) \right| < \varepsilon.$$

Finally, we would like to observe that the integral of the kernel $C(z)$ along a hyperplane is not necessarily zero; this may happen when the multiplier $\chi\,(\xi)$ is not radially symmetric. This requires a careful normalization of the function along the discontinuities. □

Proof of Corollaries 4 *and* 5. It is an immediate consequence of the previous theorems. □

Eigenfunction expansions on manifolds

In this section we would like to extend some of the results proved for Fourier integrals to Fourier series and, more generally, to eigenfunction expansion. If $\mathcal{M}$ is a smooth compact manifold of dimension d and if Δ is a second order positive elliptic operator, with smooth coefficients, selfadjoint with respect to a positive measure dx and appropriate boundary conditions, then there exist a sequence of eigenvalues $\left\{(2\pi\lambda)^2\right\}$ and a system of eigenfunctions $\{\psi_\lambda(x)\}$ orthonormal and complete in $L^2(\mathcal{M}, dx)$ and every square integrable function can be expanded in a Fourier series,

$$\langle \varphi, \psi_\lambda \rangle = \int_{\mathcal{M}} \varphi(y) \overline{\psi_\lambda(y)} dy, \qquad \varphi(x) = \sum_\lambda \langle \varphi, \psi_\lambda \rangle \psi_\lambda(x).$$

Trigonometric series on a torus and spherical harmonics expansions on a sphere are classical examples. These expansions are mean square convergent, but they may diverge pointwise and, in order to regain convergence, it may be convenient to introduce some summation methods, such as

$$\sum_\lambda \chi\left(t^{-1}\lambda\right) \langle \varphi, \psi_\lambda \rangle \psi_\lambda(x).$$

In particular, the multiplier $\chi\,(\lambda) = \left(1 - \lambda^2\right)_+^\gamma$ defines the Riesz means and, if $\gamma = 0$, the spherical sums. Pointwise convergence of eigenfunction expansions of piecewise smooth functions has been studied in [4], [5], [15], [19], [23]. In these papers it has been proved that on two-dimensional manifolds the spherical sums of eigenfunction expansions of piecewise smooth functions converge at every point, but in dimension three or higher convergence may fail even where the functions expanded are smooth. Here we consider analogous convergence problems for more general classes of functions. For simplicity in the sequel we only consider expansions in eigenfunctions of the Dirichlet problem for the Laplace operator $-\sum_j \partial^2/\partial x_j^2$ in a bounded domain $\mathcal{D}$ in $\mathbb{R}^d$ with smooth boundary. However, the results extend to other second order elliptic differential operators with appropriate boundary conditions. An integrable function on $\mathbb{R}^d$ with support on $\mathcal{D}$ can be expanded in Fourier integrals with respect to the trigonometric system $\{\exp(2\pi i\xi x)\}$ and also in Fourier series with respect to the orthonormal system $\{\psi_\lambda(x)\}$,

$$\int_{\mathbb{R}^d} \widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi = \varphi(x) \quad \text{if } x \in \mathbb{R}^d,$$
$$\sum_\lambda \langle\varphi,\psi_\lambda\rangle\psi_\lambda(x) = \varphi(x) \quad \text{if } x \in \mathcal{D}.$$

The first is an expansion in eigenfunctions of the Laplace operator in the whole space, while the second is an expansion in eigenfunctions of the Laplace operator in a bounded domain. Our goal is to compare these two expansions, under suitable smoothness assumptions of the functions expanded. As in the previous section this smoothness can be measured in terms of fractional powers of the Laplace operator, but here there are two such operators, one on $\mathbb{R}^d$ and another on $\mathcal{D}$,

$$\Delta_{\mathbb{R}^d}^{\alpha/2}\varphi(y) = \int_{\mathbb{R}^d} (2\pi|\xi|)^\alpha\, \widehat{\varphi}(\xi)\exp(2\pi i\xi y)d\xi \quad \text{if } y \in \mathbb{R}^d,$$
$$\Delta_{\mathcal{D}}^{\alpha/2}\varphi(y) = \sum_\lambda (2\pi\lambda)^\alpha\langle\varphi,\psi_\lambda\rangle\psi_\lambda(y) \quad \text{if } y \in \mathcal{D}.$$

Of course, when α is an even integer, these two operators are the same differential operator. For an arbitrary α these operators are pseudodifferential and they differ locally by a smoothing operator. Moreover, at infinity, $|\Delta_{\mathbb{R}^d}^{\alpha/2}\varphi(y)| \le c\,|y|^{-d-\alpha}$, because this is a convolution with a kernel of homogeneity $|y|^{-d-\alpha}$. In particular, if $\varphi(y)$ has support strictly inside $\mathcal{D}$, then $\Delta_{\mathcal{D}}^{\alpha/2}\varphi(y)$ is integrable in $\mathcal{D}$ if and only if $\Delta_{\mathbb{R}^d}^{\alpha/2}\varphi(y)$ is integrable in $\mathbb{R}^d$. For these reasons in the sequel we shall make no differences between these two operators. The following theorem has been suggested by some results in [4], [21] and [23]. Its content is that, under appropriate conditions, Riesz means of Fourier integrals are equiconvergent with Riesz means of expansions in eigenfunctions, that is, the means of the integrals converge if and only if the means of the series converge.

Theorem 8. *If $\varphi(y)$ is an integrable function on $\mathbb{R}^d$ with support strictly inside the domain $\mathcal{D}$ and if also $\Delta^{\alpha/2}\varphi(y)$ is an integrable function, then the Riesz means of index $\gamma \ge d-\alpha-1$ of the trigonometric Fourier integral and of the expansion in eigenfunctions are uniformly equiconvergent in every subdomain of $\mathcal{D}$ at positive distance from the boundary of $\mathcal{D}$,*

$$\lim_{t\to+\infty}\left\{\sup_{\{x\in\mathcal{D},\ d(x,\partial\mathcal{D})>\varepsilon>0\}}\left|\int_{\mathbb{R}^d}\left(1-t^{-2}|\xi|^2\right)_+^\gamma \widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi \right.\right.$$
$$\left.\left. -\sum_\lambda\left(1-t^{-2}\lambda^2\right)_+^\gamma\langle\varphi,\psi_\lambda\rangle\psi_\lambda(x)\right|\right\} = 0.$$

In particular, with the possible exception of a set of points with Hausdorff dimension $d-\alpha$ the Riesz means of index $\gamma \ge d-\alpha-1$ of the eigenfunction expansion of functions with α integrable derivatives converge,

$$\lim_{t\to+\infty}\sum_\lambda\left(1-t^{-2}\lambda^2\right)_+^\gamma\langle\varphi,\psi_\lambda\rangle\psi_\lambda(x) = \varphi(x).$$

Observe that the range of indexes $\gamma \geq d-\alpha-1$ for Riesz summability of eigenfunction expansions on manifolds is smaller than the range $\gamma > (d-1)/2-\alpha$ for Riesz summability of multiple Fourier integrals in Corollary 4. Also, observe that in the above theorem $\Delta^{\alpha/2}\varphi(y)$ is required to be an integrable function, not just a measure as in the previous section. We will see that these restrictions are necessary. The following proof is essentially taken from [4].

Proof of Theorem 8. For simplicity we write $\chi(u)$ instead of $\left(1-t^{-2}u^2\right)_+^{\gamma}$, dropping any explicit reference on γ and t. Indeed, in what follows, the precise form of $\chi(u)$ does not play any particular role. Let $\rho(u)$ be an even test function on $-\infty < u < +\infty$ with mean $\int_{-\infty}^{+\infty}\rho(u)du = 1$ and moments $\int_{-\infty}^{+\infty}\rho(u)u^j du = 0$ for $j = 1, 2, ...$ and with Fourier transform $\widehat{\rho}(s) = \int_{-\infty}^{+\infty}\rho(u)\exp(-2\pi s u)du$ supported in $\{|s| \leq 1\}$. Also, let $\zeta(u) = \varepsilon\rho(\varepsilon u)$. Then, if $\zeta * \chi(u)$ denotes the convolution $\int_{-\infty}^{+\infty}\chi(u-v)\zeta(v)dv$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\chi\left(|\xi|\right)\widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi - \sum_{\lambda}\chi(\lambda)\langle\varphi,\psi_\lambda\rangle\psi_\lambda(x)\right| \\
&\leq \left|\int_{\mathbb{R}^d}\zeta * \chi\left(|\xi|\right)\widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi - \sum_{\lambda}\zeta * \chi(\lambda)\langle\varphi,\psi_\lambda\rangle\psi_\lambda(x)\right| \\
&\quad + \left|\int_{\mathbb{R}^d}\left(\chi\left(|\xi|\right) - \zeta * \chi\left(|\xi|\right)\right)\widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi\right| \\
&\quad + \left|\sum_{\lambda}\left(\chi(\lambda) - \zeta * \chi(\lambda)\right)\langle\varphi,\psi_\lambda\rangle\psi_\lambda(x)\right|.
\end{aligned}$$

In order to show that the expansions in eigenfunction are equiconvergent with the trigonometric expansions, it suffices to prove that the first of these three terms is zero in $\{x \in \mathcal{D},\ d(x,\partial\mathcal{D}) > \varepsilon\}$ and the other two are negligible. □

Lemma 9. *If $x \in \mathcal{D}$ and $d\left(x,\partial\mathcal{D}\right) > \varepsilon$, then*

$$\int_{\mathbb{R}^d}\left(\chi\left(|\xi|\right) - \zeta * \chi\left(|\xi|\right)\right)\widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi = \sum_{\lambda}\zeta * \chi(\lambda)\langle\varphi,\psi_\lambda\rangle\psi_\lambda(x).$$

Proof of Lemma 9. By the spectral decomposition of the Laplace operator,

$$\int_{\mathbb{R}^d}\zeta * \chi\left(|\xi|\right)\widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi = \int_{-\infty}^{+\infty}\widehat{\zeta}(s)\widehat{\chi}(s)\cos\left(s\sqrt{\Delta_{\mathbb{R}^d}}\right)\varphi(x)ds,$$

with $\cos\left(s\sqrt{\Delta_{\mathbb{R}^d}}\right)\varphi(x) = \int_{\mathbb{R}^d}\cos\left(2\pi s\left|\xi\right|\right)\widehat{\varphi}(\xi)\exp(2\pi i\xi x)d\xi$ solution to the wave equation in $\mathbb{R}\times\mathbb{R}^d$ with initial position $\varphi(x)$ and initial velocity zero. Similarly,

$$\sum_{\lambda}\zeta * \chi(\lambda)\langle\varphi,\psi_\lambda\rangle\psi_\lambda(x) = \int_{-\infty}^{+\infty}\widehat{\zeta}(s)\widehat{\chi}(s)\cos\left(s\sqrt{\Delta_{\mathcal{D}}}\right)\varphi(x)ds,$$

with $\cos\left(s\sqrt{\Delta_{\mathcal{D}}}\right)\varphi(x) = \sum_{\lambda}\cos\left(2\pi s\lambda\right)\langle\varphi,\psi_\lambda\rangle\psi_\lambda(x)$ the solution to the wave equation in $\mathbb{R}\times\mathcal{D}$ with initial position $\varphi(x)$ and initial velocity zero and zero on the boundary $\partial\mathcal{D}$. Since the initial values of these two solutions agree in $\mathcal{D}$, by finite speed of wave propagation these solutions agree in $\{x \in \mathcal{D},\ |s| < d(x,\partial\mathcal{D})\}$. Since $\widehat{\zeta}(s) = \widehat{\rho}\left(\varepsilon^{-1}s\right)$ has support in $\{|s| \leq \varepsilon\}$, the lemma follows. □

Lemma 10. *If $\Delta^{\alpha/2}\varphi(y)$ is an integrable function and $\alpha+\gamma\geq d-1$, then*

$$\lim_{t\to+\infty}\int_{\mathbb{R}^d}\left|\chi\left(|\xi|\right)-\zeta*\chi\left(|\xi|\right)\right|\left|\widehat{\varphi}(\xi)\right|d\xi=0,$$
$$\lim_{t\to+\infty}\sum_{\lambda}\left|\chi(\lambda)-\zeta*\chi(\lambda)\right|\left|\langle\varphi,\psi_\lambda\rangle\right|\left|\psi_\lambda(x)\right|=0.$$

Proof of Lemma 10. The convolution $\zeta*\chi\left(u\right)$ is a good approximation of $\chi\left(u\right)$ away from the singularities of this function. More precisely, $\zeta\left(u\right)$ has integral one and other moments zero and it is concentrated in a neighborhood of the origin; moreover, in a neighborhood of the singularities $\pm t$ the function $\chi\left(u\right)$ has size $t^{-\gamma}$. This implies that $\left|\chi\left(u\right)-\zeta*\chi\left(u\right)\right|$ is essentially a sum of two bumps of height $t^{-\gamma}$ concentrated around $\pm t$,

$$\left|\chi\left(u\right)-\zeta*\chi\left(u\right)\right|\leq ct^{-\gamma}\left(1+\left|t-|u|\right|\right)^{-\eta},$$

for every η. If $\Delta^{\alpha/2}\varphi(y)$ is a finite measure, then $\left|\widehat{\varphi}(\xi)\right|\leq c\left(1+|\xi|\right)^{-\alpha}$ and

$$\begin{aligned}&\int_{\mathbb{R}^d}\left|\chi\left(|\xi|\right)-\zeta*\chi\left(|\xi|\right)\right|\left|\widehat{\varphi}(\xi)\right|d\xi\\ &\leq ct^{-\gamma}\int_{\mathbb{R}^d}\left(1+\left|t-|\xi|\right|\right)^{-\eta}\left(1+|\xi|\right)^{-\alpha}d\xi\leq ct^{d-\alpha-\gamma-1}.\end{aligned}$$

Then, if $\alpha+\gamma>d-1$ the desired estimate follows. If $\Delta^{\alpha/2}\varphi(y)$ is an integrable function, then $\lim_{|\xi|\to+\infty}|\xi|^{\alpha}\left|\widehat{\varphi}(\xi)\right|=0$ and with this extra information the desired estimate follows also in the case $\alpha+\gamma=d-1$. This proves the first part of the lemma. The proof of the second part is similar. By Cauchy inequality,

$$\begin{aligned}&\sum_{\lambda}\left|\chi\left(\lambda\right)-\zeta*\chi\left(\lambda\right)\right|\left|\langle\varphi,\psi_\lambda\rangle\right|\left|\psi_\lambda(x)\right|\\ &\leq\left\{\sum_{\lambda}\left|\chi\left(\lambda\right)-\zeta*\chi\left(\lambda\right)\right|\left|\psi_\lambda(x)\right|^2\right\}^{1/2}\left\{\sum_{\lambda}\left|\chi\left(\lambda\right)-\zeta*\chi\left(\lambda\right)\right|\left|\langle\varphi,\psi_\lambda\rangle\right|^2\right\}^{1/2}.\end{aligned}$$

By sharp estimates of the spectral function of an elliptic operator in [14], 17.5, if s is large,

$$\left\{\sum_{|\lambda-s|\leq1}\left|\psi_\lambda(y)\right|^2\right\}^{1/2}\leq cs^{d/2}\left(1+s^{1/2}d\left(y,\partial\mathcal{D}\right)\right)^{-1},$$
$$\left\{\int_{\mathcal{D}}\sum_{|\lambda-s|\leq1}\left|\psi_\lambda(y)\right|^2dy\right\}^{1/2}\leq cs^{(d-1)/2}.$$

It follows that if x is fixed and $t\to+\infty$, then

$$\left\{\sum_{\lambda}\left|\chi\left(\lambda\right)-\zeta*\chi\left(\lambda\right)\right|\left|\psi_\lambda(x)\right|^2\right\}^{1/2}\leq ct^{(d-1)/2-\gamma/2}.$$

Now observe that, by the relation $\langle \Delta^{\alpha/2}\varphi, \psi_\lambda \rangle = (2\pi\lambda)^\alpha \langle \varphi, \psi_\lambda \rangle$ and the orthonormality of the system $\{\psi_\lambda(x)\}$,

$$\left\{ \sum_{|\lambda - s| \leq 1} |\langle \varphi, \psi_\lambda \rangle|^2 \right\}^{1/2}$$

$$= \left\{ \int_{\mathcal{D}} \left| \int_{\mathcal{D}} \sum_{|\lambda - s| \leq 1} (2\pi\lambda)^{-\alpha} \psi_\lambda(x) \overline{\psi_\lambda(y)} \Delta^{\alpha/2} \varphi(y) dy \right|^2 dx \right\}^{1/2}$$

$$\leq \int_{\mathcal{D}} \left\{ \int_{\mathcal{D}} \left| \sum_{|\lambda - s| \leq 1} (2\pi\lambda)^{-\alpha} \psi_\lambda(x) \overline{\psi_\lambda(y)} \right|^2 dx \right\}^{1/2} \left| \Delta^{\alpha/2} \varphi(y) \right| dy$$

$$= \int_{\mathcal{D}} \left\{ \sum_{|\lambda - s| \leq 1} (2\pi\lambda)^{-2\alpha} |\psi_\lambda(y)|^2 \right\}^{1/2} \left| \Delta^{\alpha/2} \varphi(y) \right| dy$$

$$\leq \left\{ \sup_{d(y, \partial\mathcal{D}) \geq \delta} \sum_{|\lambda - s| \leq 1} (2\pi\lambda)^{-2\alpha} |\psi_\lambda(y)|^2 \right\}^{1/2} \int_{\mathcal{D}} \left| \Delta^{\alpha/2} \varphi(y) \right| dy$$

$$+ \left\{ \sup_{d(y, \partial\mathcal{D}) \leq \delta} \left| \Delta^{\alpha/2} \varphi(y) \right| \right\} \int_{\mathcal{D}} \left\{ \sum_{|\lambda - s| \leq 1} (2\pi\lambda)^{-2\alpha} |\psi_\lambda(y)|^2 \right\}^{1/2} dy$$

$$\leq c s^{-\alpha + (d-1)/2} \left(\int_{\mathcal{D}} |\varphi(y)| \, dy + \int_{\mathcal{D}} \left| \Delta^{\alpha/2} \varphi(y) \right| dy \right).$$

In the last inequality δ is chosen smaller than the distance of the support of $\varphi(y)$ from $\partial\mathcal{D}$ and one has to use the fact that the operator $\Delta^{\alpha/2}$ is pseudolocal and $\left|\Delta^{\alpha/2}\varphi(y)\right|$ is bounded away from the support of $\varphi(y)$. Hence, if $t \to +\infty$, then

$$\left\{ \sum_{\lambda} |\chi(\lambda) - \zeta * \chi(\lambda)| \, |\langle \varphi, \psi_\lambda \rangle|^2 \right\}^{1/2}$$

$$\leq c t^{(d-1)/2 - \gamma/2 - \alpha} \left(\int_{\mathcal{D}} |\varphi(y)| \, dy + \int_{\mathcal{D}} \left| \Delta^{\alpha/2} \varphi(y) \right| dy \right).$$

However, if $\varphi(y)$ and $\Delta^{\alpha/2}\varphi(y)$ are integrable functions, an approximation with smooth functions also gives

$$\lim_{t \to +\infty} t^{\alpha + \gamma/2 - (d-1)/2} \left\{ \sum_{\lambda} |\chi(\lambda) - \zeta * \chi(\lambda)| \, |< \varphi, \psi_\lambda >|^2 \right\}^{1/2} = 0.$$

Collecting these estimates the lemma follows. □

Observe that in Lemma 9 $\chi(u)$ does not play any role, while Lemma 10 depends on the degree of the approximation of $\chi(u)$ by $\zeta * \chi(u)$. In conclusion, the proof of Theorem 8 depends only on the degree of smoothness of $\chi(u)$ and the Riesz means can be substituted by more general operators. In particular, if $\chi(u) = \min\{1, u^\alpha\}$, then the operator associated to $\zeta * \chi(u)$ has a kernel concentrated around the diagonal $\{|x-y| \leq \varepsilon\}$, while the kernel associated to $\chi(u) - \zeta * \chi(u)$ is bounded, this is because of the fast decay $|\chi(u) - \zeta * \chi(u)| \leq c\,(1+|u|)^{-\eta}$. This proves that the operator $\Delta^{\alpha/2}$ is pseudolocal. Although the above result for general eigenfunction expansions is essentially sharp, following a suggestion by M. Taylor we give an improvement for classical multiple Fourier series on the torus. This is related to the Gauss problem of estimating the number of points with integer coordinates in large spheres. The torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ can be identified with the unit cube $[-1/2, +1/2)^d$ and an integrable function on $\mathbb{R}^d$ with support on $\mathbb{T}^d$ can be expanded both in Fourier integrals and Fourier series,

$$\int_{\mathbb{R}^d} \widehat{\varphi}(\xi) \exp(2\pi i \xi x) d\xi = \varphi(x) \quad \text{if } x \in \mathbb{R}^d,$$
$$\sum_{k \in \mathbb{Z}^d} \widehat{\varphi}(k) \exp(2\pi i k x) = \varphi(x) \quad \text{if } x \in \mathbb{T}^d.$$

As in Theorem 8 one can compare these two expansions.

Theorem 11. *If $\varphi(y)$ is an integrable function on $\mathbb{R}^d$ with support inside the unit cube $\mathbb{T}^d$ and if $\Delta^{\alpha/2}\varphi(y)$ is a finite measure, then for some $\varepsilon > 0$ the Riesz means of index $\gamma > (d - \alpha - 1 - \varepsilon)/2$ of its Fourier integral and series are equiconvergent in $\mathbb{T}^d$,*

$$\lim_{t \to +\infty} \left\{ \sup_{x \in \mathbb{T}^d} \left| \int_{\mathbb{R}^d} \left(1 - t^{-2}|\xi|^2\right)_+^\gamma \widehat{\varphi}(\xi) \exp(2\pi i \xi x) d\xi \right.\right.$$
$$\left.\left. - \sum_{k \in \mathbb{Z}^d} \left(1 - t^{-2}|k|^2\right)_+^\gamma \widehat{\varphi}(k) \exp(2\pi i k x) \right| \right\} = 0.$$

Proof of Theorem 11. The following proof is similar to the one of Theorem 8, but every step presents some small changes. We start by observing that the result for $\alpha > d$ follows from the integrability of the Fourier transform of the function expanded, while the result for $\gamma > (d-1)/2$ follows from the integrability of the Riesz kernels. Hence in the sequel we may assume $\alpha \leq d$ and $\gamma \leq (d-1)/2$. Let $\sigma(y)$ be a smooth function with $\sigma(y) = 1$ if $|y| \leq 2$ and $\sigma(y) = 0$ if $|y| \geq 3$ and define

$$\Phi(y) = \sum_{k \in \mathbb{Z}^d} \sigma\left(t^{-1}k\right) \varphi(k+y).$$

This $\Phi(y)$ is sum of a finite number of non overlapping copies of $\varphi(y)$ and these two functions agree in $\mathbb{T}^d$. As in the proof of Theorem 8, let $\chi(u) = \left(1 - t^{-2}u^2\right)_+^\gamma$, let $\rho(u)$ be an even test function with mean one and other moments zero and with

Fourier transform $\widehat{\rho}(s)$ supported in $\{|s| \leq 1\}$, also let $\zeta(u) = t\rho(tu)$. Then

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \chi(|\xi|)\, \widehat{\varphi}(\xi) \exp(2\pi i\xi x) d\xi - \sum_{k\in\mathbb{Z}^d} \chi(|k|)\, \widehat{\varphi}(k) \exp(2\pi i k x)\right| \\
&\leq \left|\int_{\mathbb{R}^d} \chi(|\xi|)\, \widehat{\varphi}(\xi) \exp(2\pi i\xi x) d\xi - \int_{\mathbb{R}^d} \chi(|\xi|)\, \widehat{\Phi}(\xi) \exp(2\pi i\xi x) d\xi\right| \\
&+ \left|\int_{\mathbb{R}^d} \zeta * \chi(|\xi|)\, \widehat{\Phi}(\xi) \exp(2\pi i\xi x) d\xi - \sum_{k\in\mathbb{Z}^d} \zeta * \chi(|k|)\, \widehat{\varphi}(k) \exp(2\pi i k x)\right| \\
&+ \left|\int_{\mathbb{R}^d} (\chi(|\xi|) - \zeta * \chi(|\xi|))\, \widehat{\Phi}(\xi) \exp(2\pi i\xi x) d\xi\right| \\
&+ \left|\sum_{k\in\mathbb{Z}^d} (\chi(|k|) - \zeta * \chi(|k|))\, \widehat{\varphi}(k) \exp(2\pi i k x)\right|.
\end{aligned}$$

Lemma 12. *If x is in $\mathbb{T}^d$ and t is large, then*

$$\int_{\mathbb{R}^d} \zeta * \chi(|\xi|)\, \widehat{\Phi}(\xi) \exp(2\pi i\xi x) d\xi = \sum_{k\in\mathbb{Z}^d} \zeta * \chi(|k|)\, \widehat{\varphi}(k) \exp(2\pi i k x).$$

Proof of Lemma 12. By the spectral decomposition of the Laplace operator,

$$\sum_{k\in\mathbb{Z}^d} \zeta * \chi(|k|)\, \widehat{\varphi}(k) \exp(2\pi i k x) = \int_{-\infty}^{+\infty} \widehat{\zeta}(s)\widehat{\chi}(s) \cos\left(s\sqrt{\Delta_{\mathbb{T}^d}}\right) \varphi(x) ds,$$
$$\int_{\mathbb{R}^d} \zeta * \chi(|\xi|)\, \widehat{\Phi}(\xi) \exp(2\pi i\xi x) d\xi = \int_{-\infty}^{+\infty} \widehat{\zeta}(s)\widehat{\chi}(s) \cos\left(s\sqrt{\Delta_{\mathbb{R}^d}}\right) \Phi(x) ds,$$

with $\cos\left(s\sqrt{\Delta_{\mathbb{T}^d}}\right) \varphi(x)$ the solution to the wave equation with periodic initial position $\sum_{k\in\mathbb{Z}^d} \varphi(k+x)$ and initial velocity zero and $\cos\left(s\sqrt{\Delta_{\mathbb{R}^d}}\right) \Phi(x)$ the solution to the wave equation with initial position $\sum_{k\in\mathbb{Z}^d} \sigma\left(t^{-1}k\right) \varphi(k+y)$ and initial velocity zero. Since the initial values of the two solutions agree in $\{|x| \leq 2t\}$, by finite speed of wave propagation these solutions agree in $\left\{x \in \mathbb{T}^d,\ |s| \leq t\right\}$. Since $\widehat{\zeta}(s)$ has support in $\{|s| \leq t\}$, the lemma follows. □

Lemma 13. *If $\alpha + 2\gamma > d - 2 + 2/(d+1)$, then*

$$\lim_{t\to+\infty} \int_{\mathbb{R}^d} |\chi(|\xi|) - \zeta * \chi(|\xi|)| \left|\widehat{\Phi}(\xi)\right| d\xi = 0,$$
$$\lim_{t\to+\infty} \sum_{k\in\mathbb{Z}^d} |\chi(|k|) - \zeta * \chi(|k|)|\, |\widehat{\varphi}(k)| = 0.$$

Proof of Lemma 13. The first observation is that $|\chi(u) - \zeta * \chi(u)|$ is essentially a sum of two bumps of height $t^{-2\gamma}$ and width t^{-1} around $\pm t$,

$$|\chi(|\xi|) - \zeta * \chi(|\xi|)| \leq ct^{-2\gamma} (1 + t\,|t - |\xi||)^{-\eta},$$

for every η. This is because $\zeta(u)$ has integral one and is concentrated in a neighborhood of radius t^{-1} of the origin and because at distance t^{-1} from the singularities in $\pm t$ the function $\chi(u)$ has size $t^{-2\gamma}$. The second observation is that when $t \to +\infty$, the annulus $\left\{||\xi|-t|\leq t^{-1}\right\}$ contains no more than $ct^{d-2+2/(d+1)}$ points with integer coordinates. This is a consequence of good estimates for the number of integer points in the spheres $\left\{|\xi|\leq t\pm t^{-1}\right\}$ in [13],

$$\left|\left|\left\{k\in\mathbb{Z}^d,\ |k|\leq t\right\}\right|-\left|\left\{\xi\in\mathbb{R}^d,\ |\xi|\leq t\right\}\right|\right|\leq ct^{d-2+2/(d+1)}.$$

The third observation is that if $\Delta^{\alpha/2}\varphi(y)$ is a finite measure, then $|\widehat{\varphi}(k)|\leq c\,|k|^{-\alpha}$. Collecting these estimates one obtains

$$\sum_{k\in\mathbb{Z}^d}|\chi(|k|)-\zeta*\chi(|k|)|\,|\widehat{\varphi}(k)|\leq ct^{d-2+2/(d+1)-2\gamma-\alpha}.$$

This proves the first estimate in the lemma. The proof of the second estimate is similar. By the Poisson summation formula,

$$\widehat{\Phi}(\xi)=\int_{\mathbb{R}^d}\left(\sum_{k\in\mathbb{Z}^d}\sigma\left(t^{-1}k\right)\varphi(k+y)\right)\exp(-2\pi i\xi y)dy$$
$$=\left(\sum_{k\in\mathbb{Z}^d}\sigma\left(t^{-1}k\right)\exp(2\pi ik\xi)\right)\widehat{\varphi}(\xi)=\left(\sum_{k\in\mathbb{Z}^d}t^d\widehat{\sigma}(t(k-\xi))\right)\widehat{\varphi}(\xi).$$

Each term of the series $\sum_{k\in\mathbb{Z}^d}t^d\widehat{\sigma}(t(k-\xi))$ has integral one, is concentrated in a ball with radius t^{-1} centered at an integer point and is negligible elsewhere. Hence, as before,

$$\int_{\mathbb{R}^d}|\chi(|\xi|)-\zeta*\chi(|\xi|)|\left|\widehat{\Phi}(\xi)\right|d\xi\leq ct^{d-2+2/(d+1)-2\gamma-\alpha}.$$

□

Lemma 14. *If $\alpha+2\gamma>d-1$, then for every x in $\mathbb{T}^d$,*

$$\lim_{t\to+\infty}\left|\int_{\mathbb{R}^d}\chi(|\xi|)\,(\widehat{\varphi-\Phi})(\xi)\exp(2\pi i\xi x)d\xi\right|=0.$$

Proof of Lemma 14. If $K(y)=\pi^{-\gamma}\Gamma(\gamma+1)\,|y|^{-\gamma-d/2}\,J_{\gamma+d/2}\,(2\pi\,|y|)$ is the convolution kernel associated to the multiplier $(1-|\xi|^2)_+^\gamma$ then, as in Theorem 1,

$$\int_{\mathbb{R}^d}\chi(|\xi|)\,(\widehat{\varphi-\Phi})(\xi)\exp(2\pi i\xi x)d\xi$$
$$=\sum_{k\in\mathbb{Z}^d-\{0\}}\sigma\left(t^{-1}k\right)\int_{\mathbb{R}^d}\Delta^{-\alpha/2}\left(t^dK(ty)\right)\Delta^{\alpha/2}\varphi(k+x-y)dy.$$

The kernel has decay $\left|\Delta^{-\alpha/2}\left(t^dK(ty)\right)\right|\leq ct^{d-\alpha}\left(1+t\,|y|\right)^{-\gamma-(d+1)/2}$, by assumptions $\int_{\mathbb{R}^d}\left|\Delta^{\alpha/2}\varphi(y)\right|dy<+\infty$ and, moreover, $\left|\Delta^{\alpha/2}\varphi(y)\right|\leq c\,|y|^{-d-\alpha}$ outside $\mathbb{T}^d$, this is because $\varphi(y)$ has support strictly inside $\mathbb{T}^d$ and $\Delta^{\alpha/2}$ is a convolution with a kernel of homogeneity $|y|^{-d-\alpha}$. Hence, if x is in $\mathbb{T}^d$ and $k\neq0$, then

$$\int_{\mathbb{R}^d}\left|\Delta^{-\alpha/2}\left(t^dK(ty)\right)\right|\left|\Delta^{\alpha/2}\varphi(k+x-y)\right|dy\leq ct^{(d-1)/2-\alpha-\gamma}\,|k|^{-\gamma-(d+1)/2}.$$

The sum of these estimates over $\{0<|k|<3t\}$ gives the desired result. □

Finally, observe that the restriction $a + 2\gamma > d - 1$ in Lemma 14 is more demanding than the restriction $\alpha + 2\gamma > d - 2 + 2/(d+1)$ in Lemma 13. Defining $\zeta(u) = t^{1-\varepsilon}\rho\left(t^{1-\varepsilon}u\right)$ and $\Phi(y) = \sum_{k\in\mathbb{Z}^d}\sigma\left(t^{\varepsilon-1}k\right)\varphi(k+y)$ with a suitable $\varepsilon > 0$, one obtains a worse Lemma 13 but a better Lemma 14 and at the end a better theorem. This concludes the proof of Theorem 11. □

Examples, counterexamples, concluding remarks

In this final section we want to compare the results obtained on euclidean spaces and on manifolds. Let us first consider the euclidean space $\mathbb{R}^d$. Corollary 4 guarantees that Riesz means with index $\gamma > (d-1)/2 - \alpha$ of functions with α integrable derivatives converge at all points where the functions are smooth. The following example shows that indeed the result is essentially sharp. The function $\varphi(y) = (1 - |y|^2)_+^{\alpha-1}$ has a bit less than α integrable derivatives and its Fourier transform has asymptotic expansion

$$\begin{aligned}&\pi^{1-\alpha}\Gamma(\alpha)\,|\xi|^{-\alpha-(d-2)/2}\,J_{\alpha+(d-2)/2}(2\pi\,|\xi|)\\ \approx\ &\pi^{-\alpha}\Gamma(\alpha)\,|\xi|^{-\alpha-(d-1)/2}\cos\left(2\pi\,|\xi| - (2\alpha+d-1)\pi/4\right).\end{aligned}$$

Hence the Riesz means of this function at the origin $x = 0$ have asymptotic expansion

$$\begin{aligned}&\pi^{1-\alpha}\Gamma(\alpha)\int_{\mathbb{R}^d}\left(1 - \left|t^{-1}\xi\right|^2\right)_+^{\gamma}|\xi|^{-\alpha-(d-2)/2}\,J_{\alpha+(d-2)/2}(2\pi\,|\xi|)d\xi\\ \approx\ &ct^{(d+1)/2-\alpha}\int_0^1\left(1-r^2\right)^{\gamma}r^{(d-1)/2-\alpha}\cos(2\pi tr - c)dr.\end{aligned}$$

The singularity of $r^{(d-1)/2-\alpha}$ when $r \to 0+$ gives to the oscillatory integral a decay $t^{\alpha-(d+1)/2}$, which compensates the factor in front of the integral, while the singularity of $\left(1-r^2\right)^{\gamma}$ when $r \to 1-$ gives a decay $t^{-\gamma-1}$ with an oscillation. Hence, the means with index $\gamma \le (d-1)/2 - \alpha$ do not converge at the origin, which is a point where the function is smooth. This phenomenon is due to the singularities of the function expanded that, propagating from the sphere $\{|y| = 1\}$, focus at the origin. This example also exhibits the speed of convergence in Theorem 2.

We now turn to the torus $\mathbb{T}^d$. In Theorem 11 we proved the Riesz summability of multiple Fourier series with $\gamma > (d - \alpha - 1 - \varepsilon)/2$. The following example for the spherical summability $\gamma = 0$ shows that the best possible ε is indeed small. Let $\delta(y) = \sum_{k\in\mathbb{Z}^d}\exp(2\pi iky)$ be the multiple Fourier series associated to the unit mass in $\mathbb{T}^d$ concentrated at zero and define

$$\varphi(y) = \Delta^{-\alpha/2}\delta(y) = \sum_{k\in\mathbb{Z}^d-\{0\}}(2\pi\,|k|)^{-\alpha}\exp(2\pi iky).$$

This function on the torus is smooth, except for a singularity at the origin of type $|y|^{\alpha-d}$. Now observe that at the point $x = (1/2, 1/2, 1/2, \ldots)$ all exponentials $\exp(2\pi ikx)$ on the spheres $\{|k| = t\}$ take the same value, $+1$ if $|k|^2 = t^2$ is even and -1 if $|k|^2 = t^2$ is odd. Hence, when grouped spherically the terms of the Fourier series have size $(2\pi t)^{-\alpha}\,|\{|k| = t\}|$. Since the number of integer points on the spheres $\{|k| = t\}$ can be of the order of t^{d-2}, for the convergence of the Fourier series one has to require at least $\alpha > d-2$. Recalling that spherical summability of Fourier integrals holds if $\alpha > (d-1)/2$, one deduces that the range of indexes for spherical summability of Fourier series is more restricted than the one for integrals.

Finally, in Theorem 8 we proved that on a compact manifold of dimension d Riesz summability holds provided that $\gamma \geq d-\alpha-1$. Despite the fact that this index is greater than the previous ones for $\mathbb{R}^d$ and $\mathbb{T}^d$, the result is sharp. The following example for spherical harmonic expansions is similar to the previous one for Fourier series. Let $\mathbb{S}^2 = \left\{x_1^2+x_2^2+x_3^2=1\right\}$ be the two-dimensional sphere in $\mathbb{R}^3$, with euclidean surface measure dx. The restriction to the sphere of the Laplace operator in space has eigenvalues $\{n(n+1)\}$, $0 \leq n < +\infty$, and an orthonormal complete system of eigenfunctions $\{\psi_{n,j}(x)\}$, $0 \leq n < +\infty$, $1 \leq j \leq 2n+1$, restriction to the sphere of harmonic homogeneous polynomials of degree n. If $P_n(z) = \dfrac{1}{2^n n!}\dfrac{d^n}{dz^n}\left(z^2-1\right)^n$ is the nth Legendre polynomial, then

$$\sum_{j=1}^{2n+1} \psi_{n,j}(x)\overline{\psi_{n,j}(y)} = \frac{2n+1}{4\pi} P_n(x\cdot y).$$

Hence the spherical harmonics expansion of a distribution on the sphere is

$$\begin{aligned}\varphi(x) &= \sum_{n=0}^{+\infty}\sum_{j=1}^{2n+1}\left[\int_{\mathbb{S}^2}\varphi(y)\overline{\psi_{n,j}(y)}dy\right]\psi_{n,j}(x)\\ &= \sum_{n=0}^{+\infty}\frac{2n+1}{4\pi}\int_{\mathbb{S}^2}P_n(x\cdot y)\varphi(y)dy.\end{aligned}$$

By Theorem 8, if $\varphi(y)$ and $\Delta^{-1/2}\varphi(y)$ are integrable functions, then the last series converges pointwise. In particular, the spherical harmonics expansion of the unit mass concentrated at x is

$$\delta(y-x) = \sum_{n=0}^{+\infty}\frac{2n+1}{4\pi}\int_{\mathbb{S}^2}P_n(y\cdot z)\delta(z-x)\,dz = \sum_{n=0}^{+\infty}\frac{2n+1}{4\pi}P_n(x\cdot y).$$

Define

$$\varphi(y) = \Delta^{-1/2}\delta(y-x) = \sum_{n=1}^{+\infty}\frac{2n+1}{4\pi\sqrt{n(n+1)}}P_n(x\cdot y).$$

Since $P_n(1) = 1$ and $P_n(-1) = (-1)^n$, the terms of the series at $\pm x$ do not converge to zero. More precisely, at the point x the series diverges to infinity and this is natural, since $\varphi(y)$ a singularity of type $|y-x|^{-1}$. At the point $-x$, where the function is smooth, the series oscillates. This phenomenon is due to the singularities of the function expanded that propagate from one point focus at the antipodal point.

In conclusion, the critical indexes for Riesz summability on euclidean spaces, on tori and spheres are all different.

The final remark is that, although up to here we were mainly concerned about functions with integrable derivatives, it is also possible and it is somehow more natural to measure smoothness using square norms. In particular, in [18], combining the Rademacher-Menchoff criterion on almost everywhere convergence of orthogonal series together with the Weyl estimates on distribution of eigenvalues, it is shown that the eigenfunction expansions of functions in Sobolev classes $(1+\Delta)^{-\alpha/2}L^2(\mathcal{M},dx)$ with $\alpha > 0$ converge almost everywhere. Following [2] and [22], it is also possible to estimate the capacity of the divergence sets. The following theorem and its proof are essentially a mix of ideas in [18] and [17].

Theorem 15. *Let us introduce the Bessel potentials,*

$$(1+\Delta)^{-\alpha/2}\,\mu(x)=\sum_{\lambda}\left(1+(2\pi\lambda)^2\right)^{-\alpha/2}\langle\mu,\psi_\lambda\rangle\psi_\lambda(x).$$

Moreover, let us define the α capacity of a set X in $\mathcal{M}$,

$$C_\alpha(X)=\inf\left\{\int_{\mathcal{M}}|\mu(x)|^2\,dx,\ \textit{with}\ (1+\Delta)^{-\alpha/2}\,\mu(x)\geq 1\ \textit{in}\ X\right\}.$$

Then, if $\alpha>0$,

$$\begin{aligned}&C_\alpha\left(\left\{x\in\mathcal{M},\ \sup_{t>0}\left|\sum_{\lambda<t}\langle\varphi,\psi_\lambda\rangle\psi_\lambda(x)\right|>\lambda\right\}\right)\\&\leq c\lambda^{-2}\sum_{\lambda}\log^2\left(2+(2\pi\lambda)^2\right)\left(1+(2\pi\lambda)^2\right)^{\alpha}|\langle\varphi,\psi_\lambda\rangle|^2.\end{aligned}$$

In particular, if $\log(2+\Delta)\,(1+\Delta)^{\alpha/2}\,\varphi(x)$ is square integrable, then the α capacity of the set where its expansion in eigenfunctions diverges is zero,

$$C_\alpha\left(\left\{x\in\mathcal{M},\ \lim_{t\to+\infty}\sum_{\lambda<t}\langle\varphi,\psi_\lambda\rangle\psi_\lambda(x)\neq\varphi(x)\right\}\right)=0.$$

Proof of Theorem 15. Let $\varphi(x)=(1+\Delta)^{-\alpha/2}\,\mu(x)$. By synthesizing the potentials via fundamental solutions to the heat equation, one can see that the kernels associated to the operators $(1+\Delta)^{-\alpha/2}$ with $\alpha>0$ are positive. This implies that

$$\begin{aligned}&\sup_{t>0}\left|\sum_{\lambda<t}\langle(1+\Delta)^{-\alpha/2}\,\mu,\psi_\lambda\rangle\psi_\lambda(x)\right|\\&\qquad\leq(1+\Delta)^{-\alpha/2}\left(\sup_{t>0}\left|\sum_{\lambda<t}\langle\mu,\psi_\lambda\rangle\psi_\lambda\right|\right)(x).\end{aligned}$$

Moreover, it follows from the Rademacher-Menchoff theorem, [27], 13.10, together with the Weyl estimates on distribution of eigenvalues, [14], 17.5, that

$$\int_{\mathcal{M}}\left(\sup_{t>0}\left|\sum_{\lambda<t}\langle\mu,\psi_\lambda\rangle\psi_\lambda(x)\right|\right)^2dx\leq c\sum_{\lambda}\log^2\left(2+(2\pi\lambda)^2\right)|\langle\mu,\psi_\lambda\rangle|^2.$$

Hence, if $\varphi(x)=(1+\Delta)^{-\alpha/2}\,\mu(x)$, then

$$\begin{aligned}&C_\alpha\left(\left\{x\in\mathcal{M},\ \sup_{t>0}\left|\sum_{\lambda<t}\langle\varphi,\psi_\lambda\rangle\psi_\lambda(x)\right|>\lambda\right\}\right)\\&\leq C_\alpha\left(\left\{x\in\mathcal{M},\ (1+\Delta)^{-\alpha/2}\left(\sup_{t>0}\left|\sum_{\lambda<t}\langle\mu,\psi_\lambda\rangle\psi_\lambda\right|\right)(x)>\lambda\right\}\right)\\&\leq\lambda^{-2}\int_{\mathcal{M}}\left(\sup_{t>0}\left|\sum_{\lambda<t}\langle\mu,\psi_\lambda\rangle\psi_\lambda(x)\right|\right)^2dx\\&\leq c\lambda^{-2}\sum_{\lambda}\log^2\left(2+(2\pi\lambda)^2\right)\left(1+(2\pi\lambda)^2\right)^{\alpha}|\langle\varphi,\psi_\lambda\rangle|^2.\end{aligned}$$

Finally, the pointwise convergence of the eigenfunction expansion follows from the boundedness of the maximal partial sums by a standard argument. ☐

Observe that the above result holds for every eigenfunction expansion and it is independent from the underlying manifold.

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Dipartimento di Matematica, Università di Milano–Bicocca, Edificio U5, via R.Cozzi 53, 20125 Milano, Italia
E-mail address: leonardo@matapp.unimib.it